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'Inactive' motion and pressure fluctuations in turbulent boundary layers

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Townsend's (1961) hypothesis that the turbulent motion in the inner region of a boundary layer consists of (i) an 'active' part which produces the shear stress τ and whose statistical properties are universal functions of τ and y, and (ii) an 'inactive' and effectively irrotational part determined by the turbulence in the outer layer, is supported in the present paper by measurements of frequency spectra in a strongly retarded boundary layer, in which the 'inactive' motion is particularly intense. The only noticeable effect of the inactive motion is an increased dissipation of kinetic energy into heat in the viscous sublayer, supplied by turbulent energy diffusion from the outer layer towards the surface. The required diffusion is of the right order of magnitude to explain the non-universal values of the triple products measured near the surface, which can therefore be reconciled with universality of the 'active' motion.

Dimensional analysis shows that the contribution of the 'active' inner layer motion to the one-dimensional wave-number spectrum of the surface pressure fluctuations varies as τ_w^2/k_1 up to a wave-number inversely proportional to the thickness of the viscous sublayer. This result is strongly supported by the recent measurements of Hodgson (1967), made with a much smaller ratio of microphone diameter to boundary-layer thickness than has been achieved previously. The disagreement of the result with most other measurements is attributed to inadequate transducer resolution in the other experiments.

1. Introduction

The pressure fluctuation at a point in or near a turbulent flow is determined by a volume integral over the whole turbulent region, and irrotational velocity fluctuations will be induced both outside the turbulent region (where they are easy to identify (Phillips 1955; Bradshaw 1967*b*)) *and* inside it (where they are superimposed on the vorticity field). The general theory of decomposition of a turbulent flow into separate but interacting modes is given by Chu & Kovasznay (1958), but quantitative discussion cannot be carried very far unless the length scale of the irrotational field is locally much larger than the length scale of the vorticity field so that the latter is passively convected by the former. However, with this proviso about length scales even two *vorticity* fields would not interact, because of the independence of Fourier components for distant wave-numbers (Batchelor 1953, p. 109), so that the larger-scale vorticity field could be regarded as irrotational when discussing the smaller, providing that the dissipation rate in the large-scale field was negligible.†

In the present paper it is shown that the turbulent motion in the inner layer of a turbulent wall flow ($y/\delta < 0{\cdot}2$, say) can be divided into two essentially independent parts, respectively the vorticity field of the inner layer proper and a larger-scale 'inactive' motion arising in the outer layer: this latter motion is partly the true irrotational field associated with pressure fluctuations generated in the outer layer, and partly the large-scale vorticity field of the outer-layer turbulence which the inner layer sees as an unsteady external stream. The vorticity field of the inner layer proper is universal, with velocity scale $(\tau/\rho)^{\frac{1}{2}}$ and length scale y.

Measurements of pressure fluctuations at the surface can be used to derive some useful information about the irrotational fluctuations within the turbulent flow and are also of practical importance because of structural excitation and consequent noise radiation. In this paper, measurements of surface pressure fluctuations below a strongly-retarded equilibrium boundary layer (Bradshaw 1967*c*) are presented and used in the discussion of the inner-layer turbulence mentioned in the last paragraph. In turn, the clarification of our ideas about the universality of the inner-layer turbulence is used to derive a universal form for the high-wave-number part of the pressure-fluctuation spectrum.

The immediate practical application of this work is to the development of a method of calculating boundary-layer development by transforming the turbulent kinetic energy equation into an equation for shear stress (Bradshaw, Ferriss & Atwell 1967). The transformation is more straightforward and more convincing when the behaviour of the inactive motion is taken into account, because it is possible to treat the energy equations for the active and inactive motions separately—or in practice to ignore the latter altogether—so that the universal features of the shear-stress-producing turbulence are not obscured by the inactive motion near the surface or the irrotational field at the edge of the boundary layer.

'Inactive' motion will also occur in the inner layer of the Earth's boundary layer, so that the low-wave-number spectra will depend on the turbulence at altitude as well as on the surface shear stress and heat flux: this may explain some of the scatter between different spectra measured with nominally identical surface conditions.

2. Motion in the inner layer

Near the surface but outside the viscous sublayer, the only important length scale is y (normal distance from the wall), and the only important velocity scale is $u_\tau \equiv (\tau_w/\rho)^{\frac{1}{2}}$, or, as a better approximation taking account of the variation of shear stress with y, $(\tau/\rho)^{\frac{1}{2}}$. Thus, immediately, $\partial U/\partial y = (\tau/\rho)^{\frac{1}{2}}/Ky$, where K is von Kármán's constant, 0·40 approximately, and moreover $\phi(k_1) = (\tau/\rho)\, y f(k_1 y)$ where $\phi(k_1)$ is the wave-number spectral density of any fluctuating velocity and k_1 is the component of the wave-number vector $\mathbf{k}$ in the flow direction (x).

† For instance, the aircraft designer usually ignores the meteorologist's vorticity field, although the velocity fluctuations are of the same order in the two cases.

Now although the above expression connecting velocity gradient and turbulent shear stress in the inner layer (the mixing length formula) has received a great deal of experimental support and has been carefully re-appraised by Townsend (1961), it is—as Townsend comments—well known that the rest of the turbulent motion does *not* scale on τ and y 'and it is difficult to reconcile these observations

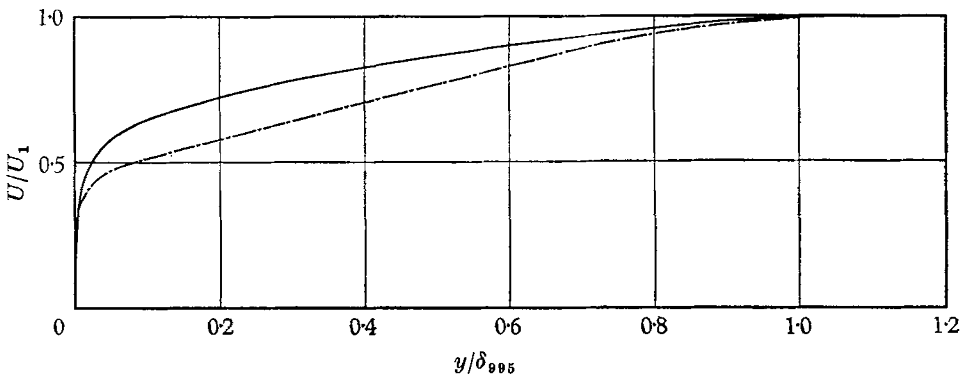


FIGURE 1. Mean velocity profiles at $x = 83$ in. —, U_1 = constant, $U_1\delta_1/\nu = 13200$, $\delta_{995}/\delta_1 = 6{\cdot}02$, $\delta_{995} = 1{\cdot}25$ in.; —·—·, $U_1 \propto x^{-0{\cdot}255}$, $U_1\delta_1/\nu = 60500$, $\delta_{995}/\delta_1 = 4{\cdot}11$, $\delta_{995} = 4{\cdot}5$ in.

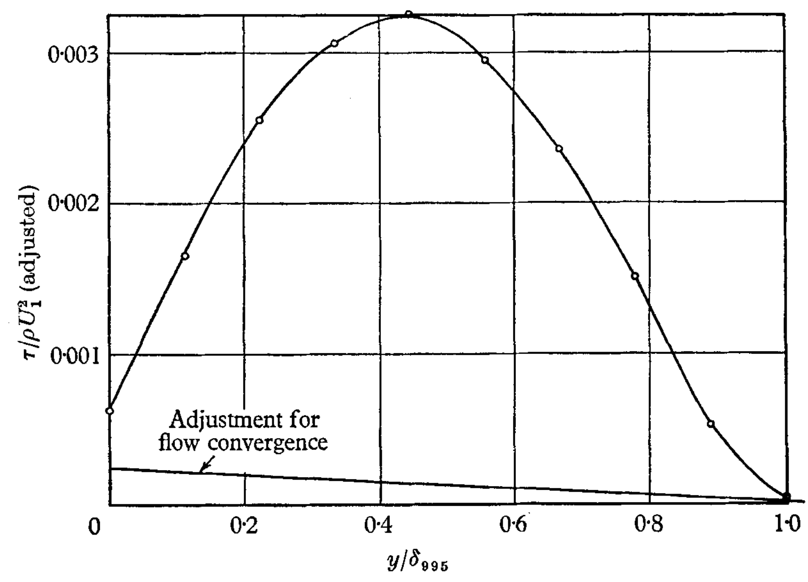


FIGURE 2. Shear stress profile at $x = 83$ in. $U_1 \propto x^{-0{\cdot}255}$.

without supposing that the motion at any point consists of two components, an active component responsible for turbulent transfer and determined by the stress distribution, and an inactive component which does not transfer momentum or interact with the universal component' (indeed, the only other course would be to believe that the scales of the mean motion were universal but that the scales of the shear-stress-producing turbulence were not—a sentiment open to doubt).

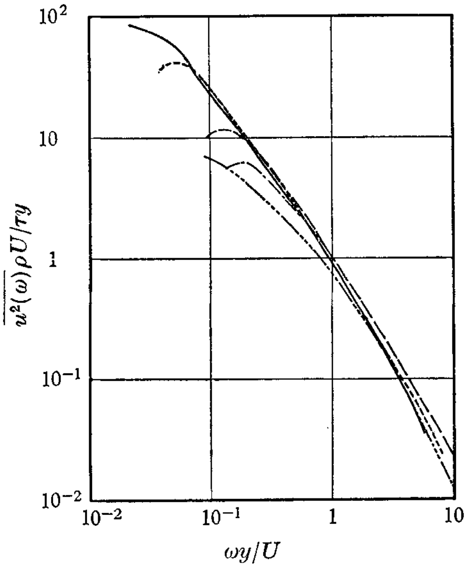


FIGURE 3*a*. Inner-layer frequency spectra at x = 83 in., u component.

y/δ_{995} = 0·022, = 0·044, = 0·11, = 0·18 } $a = -0{\cdot}255$
y/δ = 0·2 } $a = 0$

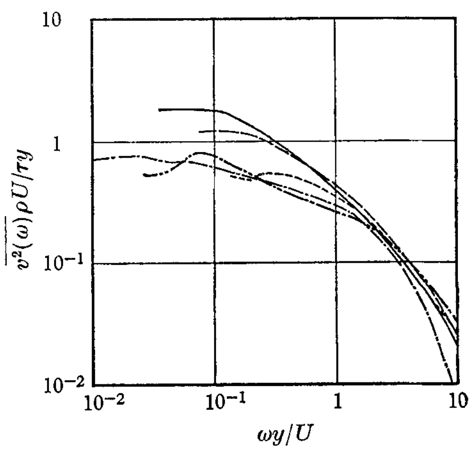


FIGURE 3*b*. Inner-layer frequency spectra at x = 83 in., v component.

y/δ_{995} = 0·055, = 0·11, = 0·22 } $a = -0{\cdot}255$,
= 0·06, = 0·20 } $a = 0$.

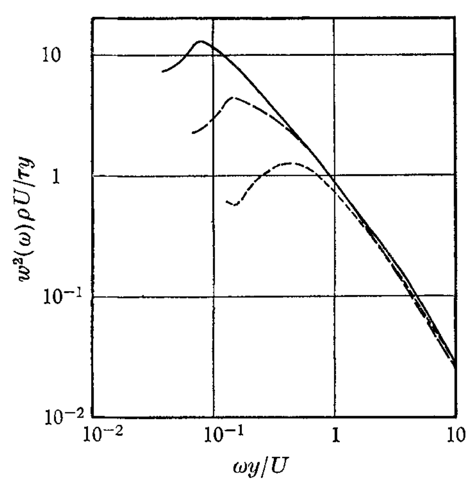


FIGURE 3*c*. Inner-layer frequency spectra at x = 83 in., w component.

y/δ_{995} = 0·055, = 0·11, = 0·22 } $a = -0{\cdot}255$.

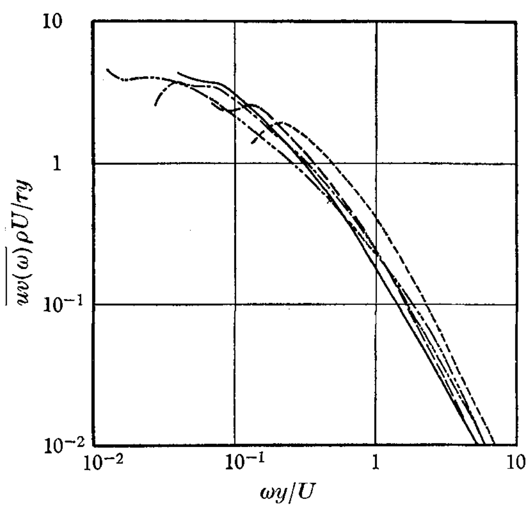


FIGURE 3*d*. Inner-layer frequency spectra at x = 83 in., uv.

y/δ_{995} = 0·031, = 0·055, = 0·11, = 0·22 } $a = -0{\cdot}255$,
y/δ = 0·085} $a = 0$.

Townsend goes on to suggest that 'the inactive motion is a meandering or swirling motion made up from attached eddies of large size which contribute to the Reynolds stress much farther from the wall than the points of observation' (but not *at* the point of observation).

Figures 3 (*a*) to 3 (*d*) show frequency spectra of $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ (where (u, v, w) are the components of the velocity fluctuation in the (x, y, z) directions) and $\overline{uv}$ in the inner layers of boundary layers with $U_1 \propto x^a$, for $a = 0$ and $a = -0{\cdot}255$ (see figures 1 and 2). The measurements were made in the course of an investigation of 'equilibrium' (self-preserving) boundary layers. Full details of the investigation and of the experimental arrangements are given by Bradshaw (1967*c*). Casual readers can treat the present paper as self-contained, and indeed the only relevance of the fact that these are equilibrium boundary layers is that the differences in x-position (shown in the figures for the sake of consistency with Bradshaw (1967*c*)) can be ignored in the present discussion. The boundary layers with $a = 0$ and $a = -0{\cdot}255$ have weak and strong 'inactive' motion respectively. The spectrum results show the scatter inevitable with hot-wire measurements, but it is clear that to a fair approximation the $\overline{uv}$ spectra near the surface are similar at all wave-numbers not directly affected by viscosity, the $\overline{v^2}$ spectrum shows departures from similarity for $\omega y/U \sim k_1 y < 2$ (where ω is frequency, in rad./sec), and the $\overline{u^2}$ and $\overline{w^2}$ spectra are grossly dissimilar for $k_1 y < 2$. The low-wave-number parts of the $\overline{u^2}$ and $\overline{w^2}$ (but not the $\overline{v^2}$) spectra in the retarded boundary layer collapse quite well when made dimensionless with *constant* scales such as U_1 and δ. The motion at the highest wave-numbers is directly affected by viscosity (e.g. figure 3 (*b*) $a = 0$, $y/\delta = 0{\cdot}06$), and is not expected to plot on the inner layer scales: the present measurements collapse well on the Kolmogorov local-isotropy variables (Bradshaw 1967*a*). The contribution of this range of wave-numbers to the shear stress is negligible, by definition, outside the viscous sublayer. Taking the lowest wave-number affected by viscosity to be $0{\cdot}1\ (\epsilon/\nu^3)^{\frac{1}{4}}$ and noting that $\epsilon = (\tau/\rho)^{\frac{3}{2}}/Ky$, (Townsend 1961; Bradshaw *et al.* 1967) we see that at $(\tau/\rho)^{\frac{1}{2}} y/\nu = 40$ viscous effects extend down to $k_1 y = 2$: this value of $(\tau/\rho)^{\frac{1}{2}} y/\nu$ is usually quoted as that at which the Reynolds stress starts to decrease, and indeed figure 3 (*d*) implies that $\overline{uv}$ will start to decrease appreciably once viscous effects extend down to $k_1 y \approx 2$. More detailed results for $(\tau/\rho)^{\frac{1}{2}} y/\nu < 40$ are given by Clark (1966).

If the shear-stress-producing motion in the inner layer is indeed universal, then it cannot be directly influenced by the 'large eddies' (Townsend 1956), which have scales typical of the outer layer. Also, the suggestion made by several workers that the fully turbulent flow is controlled by disturbances originating in the viscous sublayer is extremely difficult to reconcile with inner layer universal scaling, but probably the strongest argument against this suggestion is the empirical fact that von Kármán's constant is the same for smooth and rough walls, although the viscous sublayer as such does not exist in the latter case.

Another useful piece of information that can be deduced from figures 3 (*a*) to 3 (*d*) is that inner layer universal scaling is valid even if τ/ρ varies considerably across the inner layer (see figure 2): this justifies its use in compressible flow (Bradshaw & Ferriss 1966) where ρ rather than τ varies across the inner layer.

Since the non-universal component does not produce much shear stress or disturb the universality of the smaller-scale motion, it fits Townsend's description, above, and we shall now call it 'inactive'. That it does not interact with the smaller-scale motion follows from the large difference in scales; but its failure to contribute to the shear stress in the inner layer cannot be explained in the same way, because the large-scale motion in the outer region certainly produces shear stress. The explanation is partly that the v component of any sort of large-scale motion is small at distances from the wall small compared with a longitudinal wavelength, as follows from the continuity equation: however, the uv spectra are much more nearly universal at low frequencies than the v spectra, so that for a full explanation we must identify the source of the inactive motion.

It is not possible to isolate the inactive motion, but presumably its spectral density is at least as large as the difference between the spectra in the retarded boundary layer and in the constant-pressure boundary layer, and its intensity at least as large as the difference in intensity in the two boundary layers for a given value of shear stress. One obvious source is the pressure fluctuations generated by the turbulence in the outer layer, and on the strength of the pressure-velocity correlations (figure 7 (a)) we may attribute at least $(0{\cdot}3)^2 \sim 0{\cdot}1$ of the u-component mean-square intensity near the wall in the retarded boundary layer to this cause: however, a comparison of the intensity measurements in this boundary layer (Bradshaw 1967c) with Klebanoff's (1955) measurements in zero gradient indicate that this is only 0·2–0·3 of the intensity of the 'inactive' component. The rest of the 'inactive' motion is caused partly by incursions of fluid from the outer layer, which will be small because the v component of the inactive motion is small, and partly by the unsteady external stream seen by the inner layer: either phenomenon will produce a slow fluctuation of the (quasi-steady) universal inner layer flow which, it will be now shown, is not likely to show up in mean-flow measurements. For simplicity we assume that the shear-stress gradient is small, so that the velocity profile in the inner layer is logarithmic in the absence of inactive motion,

$$\frac{U}{(\tau/\rho)^{\frac{1}{2}}} = \frac{1}{K}\left[\log\frac{(\tau/\rho)^{\frac{1}{2}}y}{\nu} + A\right].$$

Let us assume that the shear stress fluctuates slowly and sinusoidally in time, so that we can write

$$\tau' = \tau(1+\alpha\cos\omega t),$$

where ω is small compared to a typical frequency ω_a of the 'active' motion. Now we can define short-term averages, denoted by τ', U', etc., by averaging over a time T_1 such that $1/\omega_a \ll T_1 \ll 1/\omega$ and also define the usual long-term averages, τ, U, etc., by averaging over a time $T_2 \gg 1/\omega$. Assuming that α is small, we get

$$((\tau/\rho)^{\frac{1}{2}})' = (\tau/\rho)^{\frac{1}{2}}(1+\tfrac{1}{2}\alpha\cos\omega t - \tfrac{1}{8}\alpha^2\cos^2\omega t + O(\alpha^3)).$$

Assuming that the logarithmic velocity profile holds for the short-term-average quantities U' and τ' (i.e. assuming quasi-steady flow) we find that the long-term-average velocity profile is

$$\frac{U}{(\tau/\rho)^{\frac{1}{2}}} = K\left[\left(1-\frac{\alpha^2}{16}\right)\left(\log\frac{(\tau/\rho)^{\frac{1}{2}}y}{\nu}+A\right)+O(\alpha^3)\right],$$

and also that the mean-square 'inactive' velocity fluctuation is

$$\frac{\overline{(U'-U)^2}}{\tau/\rho} = \frac{\alpha^2}{8}\left(\frac{U}{(\tau/\rho)^{\frac{1}{2}}}+\frac{1}{K}\right)^2 + O(\alpha^3).$$

If we assume, for example, that the root-mean-square 'inactive' fluctuation is 20% of the local mean velocity we find that the root-mean-square fluctuation in shear stress is 40% ($\alpha = 0{\cdot}56$) but the difference between the mean velocity and the mean velocity in a steady flow with the same average shear stress is only about 2% (taking U/u_τ to be about 15).

Therefore very large fluctuations can be imposed on the inner layer without noticeable effect on the mean quantities (for a similar conclusion based on experimental evidence, see Karlsson (1959)). In particular, large fluctuations of shear stress can occur, and it is only in the mean that we can call the large-scale motion 'inactive' in the simplest sense of Townsend's hypothesis.

3. The energy balance for the inactive motion

Since the inactive motion is of large scale and does not contribute to the Reynolds shear stress it does not directly produce or dissipate any significant amount of turbulent energy. In the quasi-steady oscillating inner layer, local production and dissipation are nominally equal, according to the crude analysis above, both being

$$\overline{(\tau'/\rho)^{\frac{3}{2}}}/Ky \quad \text{or} \quad (1+\tfrac{3}{16}\alpha^2)(\overline{\tau}/\rho)^{\frac{3}{2}}/Ky.$$

If we take $\alpha = 0{\cdot}56$ as before, the dissipation is only 6% greater than in an ordinary inner layer with the same mean shear stress, which is barely noticeable. However, this equality does not extend into the viscous sublayer, where direct dissipation of quasi-mean flow energy into heat takes place, so that for this reason alone we may expect a flow of energy of the inactive motion towards the wall, equal to the additional rate of direct dissipation caused by the oscillations. This energy flux is carried by eddies with wave-numbers typical of the outer motion: the local production of turbulent energy takes place among eddies with wave-numbers typical of the 'active' inner motion and cannot supply energy for direct dissipation of quasi-mean flow energy. The rate of direct dissipation is $\nu(\partial U/\partial y)^2$ and the integral of this is

$$\nu\int_{y=0}^{\infty}\frac{dU}{dy}\,dU.$$

Using the implicit empirical formula for the inner layer suggested by Burton (1965),

$$u_\tau y/\nu = U/u_\tau + (U/8{\cdot}74u_\tau)^7,$$

where $u_\tau = (\tau/\rho)^{\frac{1}{2}}$, the total extra rate of direct dissipation is $1{\cdot}7\alpha^2(\overline{\tau}/\rho)^{\frac{3}{2}}$ or $0{\cdot}54(\overline{\tau}/\rho)^{\frac{3}{2}}$ with the same value of α as above. To supply this energy by diffusion, we expect that $\overline{pv}/\rho+\frac{1}{2}\overline{q^2v}$ will tend to the value $-0{\cdot}54\ (\overline{\tau}/\rho)^{\frac{3}{2}}$ in the inner layer and go sharply to zero at the wall. In fact $\frac{1}{2}\overline{q^2v}$ is about $-1{\cdot}0\ (\overline{\tau}/\rho)^{\frac{3}{2}}$, or rather less, in the inner layer of the retarded boundary layer with $a = -0{\cdot}255$, compared with about $+0{\cdot}1\ (\overline{\tau}/\rho)^{\frac{3}{2}}$ (zero to the likely accuracy of the measurements) in the constant-pressure boundary layer (figures 4 (*a*) and 4 (*b*)), which certainly implies extra dissipation, concentrated very near the wall and not merely distributed over the inner layer, and qualitatively supports the crude analysis for the oscillating inner layer.

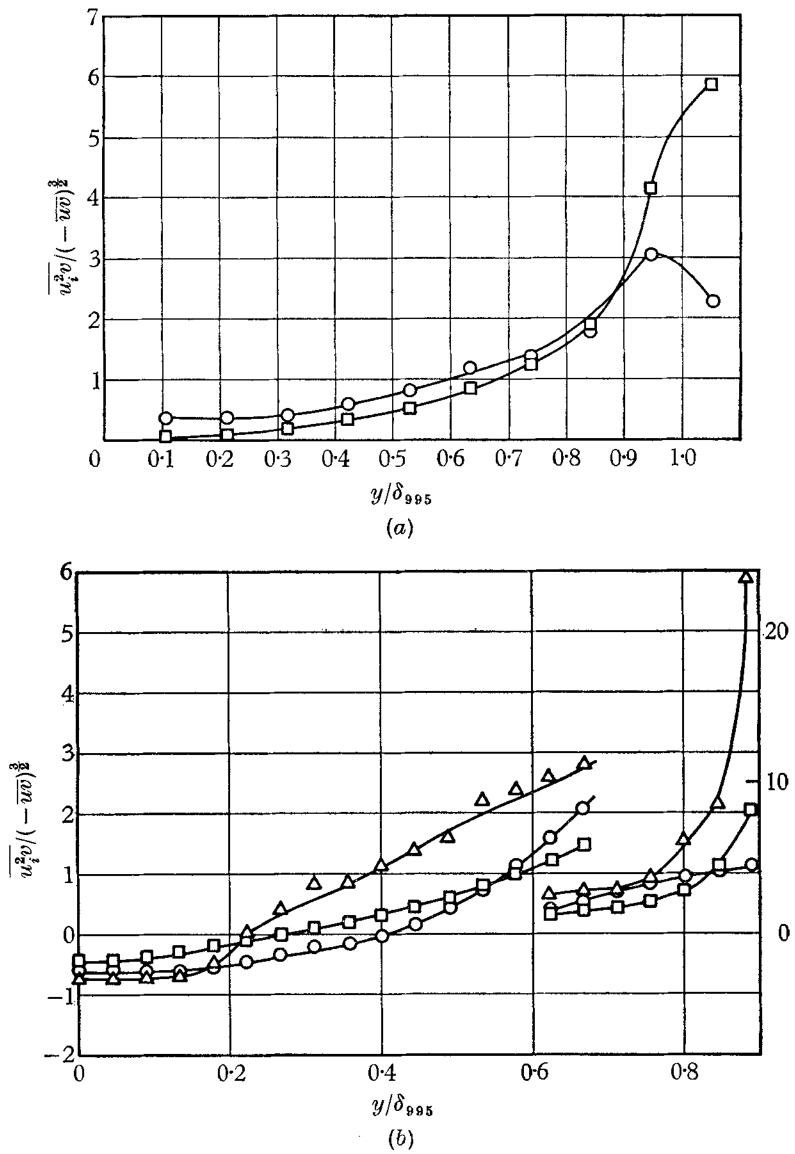


FIGURE 4. Components of $\overline{q^2 v}$ at x = 83 in. (*a*) zero pressure gradient, (*b*) $U_1 \propto x^{-0\cdot255}$. $a = -0\cdot255$. ○, $\overline{u^2 v}$; □, $\overline{v^3}$; △, $\overline{vw^2}$.

Townsend (1961) argued from considerations of inner-layer universality that $(\overline{pv}/\rho + \frac{1}{2}\overline{q^2 v})/(\tau/\rho)^{\frac{3}{2}}$ should take a universal constant value near the wall: clearly we must regard this as the value for the active motion only, and it seems very unlikely that it is as large numerically as $-0\cdot3$ (deduced by Townsend from the

Pitot profiles of Stratford (1959) in a continuously separating, and therefore highly turbulent, boundary layer), since such a 'universal' value can hardly be reconciled with the direct measurements of $\overline{q^2v}$ in zero pressure gradient where the inactive motion is not very strong. Reynolds's (1965) estimates, varying from about -2 to -3 for different types of shear flow, were obtained by a process amounting to triple differentiation of mean velocity profiles, which seems to be placing undue confidence in the behaviour of Pitot tubes in turbulent flow. The hot-wire measurements themselves are rather difficult and not very reliable since the response, even of linearized hot wires, in highly turbulent flows is suspect; for instance, it is plausible that *most* of the extra energy lost directly to heat at the wall should be supplied by $\overline{u^2v}$ but according to the measurements $\overline{vw^2}$ is larger. It should be noted that no confidence can be placed in previous published measurements of triple products made with unlinearized wires.

Apart from the effects of the extra dissipation in the viscous sublayer we expect the energy balance for the inactive motion to reduce to 'advection = diffusion', or

$$\tfrac{1}{2}(U\,\partial\overline{q^2}/\partial x + V\,\partial\overline{q^2}/\partial y) = -(\partial/\partial y)(\overline{pv}/\rho + \tfrac{1}{2}\overline{q^2v}),$$

where it is understood that the symbols refer to the 'inactive' contribution only. It is qualitatively clear that much of the difference between the measurements of $\overline{q^2v}$ in the two boundary layers (and of course *all* of the difference in the inner layer) can be attributed to the 'inactive' motion, and the same is probably true of the unmeasured $\overline{pv}$.

The energy of the 'inactive' motion near the surface and near the outer edge is diffused from the active motion in the high-intensity region of the layer, and this loss of energy of the active motion must be taken account of in the energy balance for the active motion. The extra direct dissipation in the viscous sublayer of the boundary layer with $U_1 \propto x^{-0.255}$ is only about 1 % of the total production in the high-intensity part of the layer and the advection near the wall is also small so that the chief sink of energy in the inactive motion is the advection near the outer edge of the flow.

4. Surface pressure fluctuations in the strongly retarded boundary layer

(*a*) *The convection velocity*

In boundary layers or jets the turbulent intensity in the region where most of the pressure fluctuations are generated is high enough for the velocity-fluctuation intensity to be spread over an appreciable band of phase velocities ω/k_1 at a given wave-number k_1, so that it is not possible to assign a single value to the convection velocity even at a given distance from the surface. Since the surface pressure fluctuation is obtained as an integral over the whole thickness of the boundary layer, it is spread over an even wider band of phase velocities, and the centroid of this band is a rapidly varying function of wave-number. Wills (1967) has pointed out the advantages of presenting results in the $(k_1, \omega/k_1)$ or (k_1, ω)-planes rather than the (r_1, τ)-plane ('space-time correlations') and has deduced the $(k_1, \omega/k_1)$ spectrum of surface pressure fluctuations in zero pressure gradient by taking the one-dimensional Fourier transform of frequency-filtered space correlations. We have made similar measurements in the retarded equilibrium boundary layer, with $U_1 \propto x^{-0.255}$, which are shown in figure 5. They have been normalized so that the integral over all wave-numbers at any given frequency is equal to the dimensionless frequency spectral density measured with a $\frac{1}{8}$ in. (0·32 cm.) diameter transducer (figure 6): the smaller hot-wire-orifice transducer (Wills 1965)

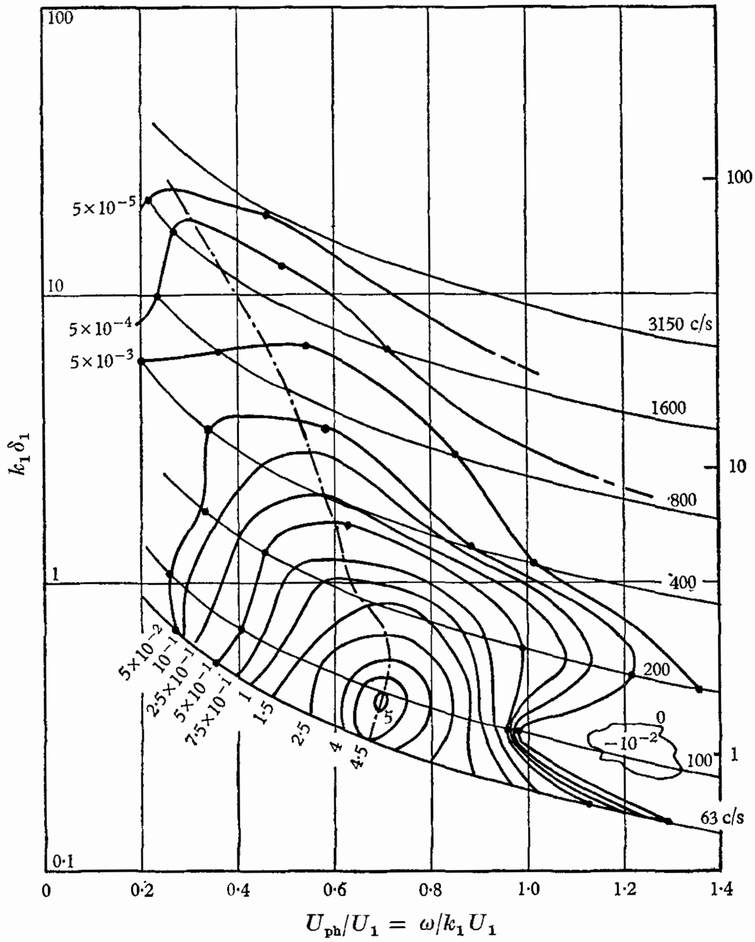


FIGURE 5. Spectral density of surface pressure fluctuations: contours in phase-velocity/wave-number plane at x = 48 in. $U_1 \propto x^{-0.255}$.

used to measure the frequency-filtered space correlations does not have a flat frequency response and is less suitable for spectrum measurements. The values of spectral density attached to the contours become increasingly unreliable as the wave-number increases because of lack of spatial resolution of the $\frac{1}{8}$ in. transducer.

The intuitively obvious definition of the average convection velocity at a given wave-number is that given by the centroid of a cross-section of figure 5 at constant wave-number, and is tabulated as $\overline{U}_{ph}$ in table 1: this is generally a little lower than the phase velocity at which the spectral density is a maximum (this latter is the definition used by Wills, and is closely related to the velocity of the frame of reference in which time derivatives are a minimum, which is the appropriate definition in problems of wave generation). The convection velocity so defined is everywhere significantly less than in zero pressure gradient, and the overall convection velocity integrated over all wave-numbers is $0{\cdot}65U_1$ compared with the

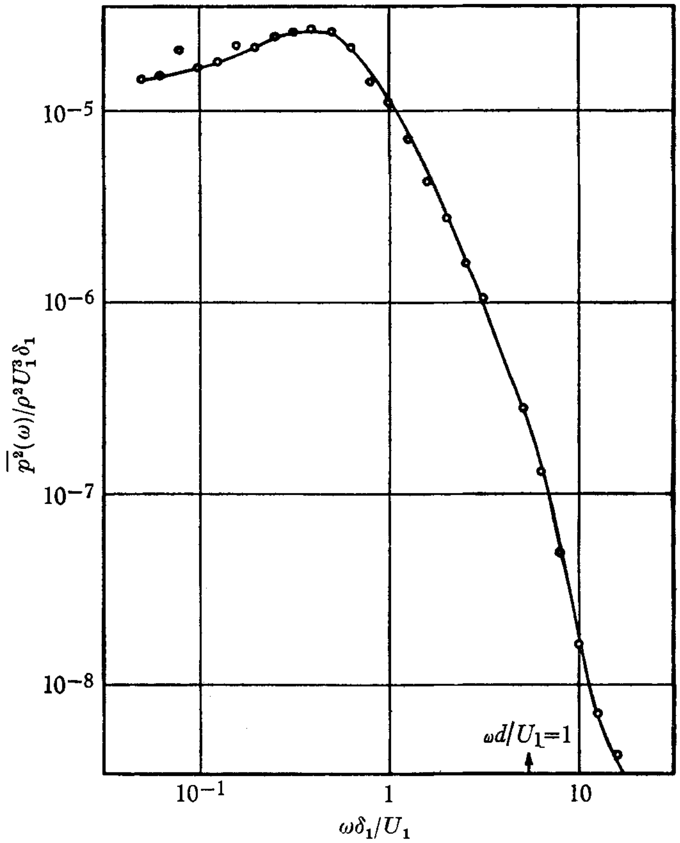


FIGURE 6. Frequency spectrum of surface pressure fluctuation at $x = 48$ in. $U_1 \propto x^{-0{\cdot}255}$.

$k_1\delta_1$	0·4	0·5	0·6	0·8	1·0	2·0	5·0	10·0
$\overline{U}_{ph}/U_1$	0·662	0·679	0·695	0·705	0·658	0·540	0·451	0·422
k_1y_k	0·54	0·71	0·98	1·33	1·38	1·17	0·79	0·97
σ/U_1	0·16	0·15	0·16	0·17	0·17	0·14	0·16	0·13

TABLE 1. Convection velocity of surface pressure fluctuations in the strongly retarded boundary layer

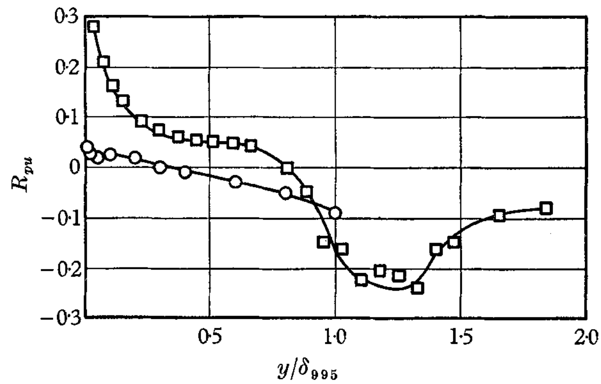
generally accepted value of about $0{\cdot}8U_1$ in zero pressure gradient. It is qualitatively obvious that most of the turbulent energy in the retarded boundary layer resides in the part of the layer where the mean velocity of the fluid is $0{\cdot}6$–$0{\cdot}7U_1$, but the high-wave-number contribution from the motion in the inner layer might, if adequately recorded by the pressure transducer, reduce the overall convection velocity considerably in both boundary layers (see § 4 (*d*)) because the convection velocity decreases as the wave-number increases. If $\bar{y}_k$ is the distance from the surface at which the mean velocity is equal to the convection velocity for a given wave-number k_1, then $k_1\bar{y}_k \approx 1$ for $k_1\delta_1 \geqslant 1$ (values of $\bar{y}_k$ given by Bradshaw (1965) are incorrect). For $k_1\delta_1 < 1$, $\bar{y}_k$ becomes more nearly constant: it is then in the region of maximum turbulent intensity. A constant value of $k_1\bar{y}_k$ would be expected if the surface pressure fluctuations were generated by a 'universal' turbulent motion whose spectra scaled on the dimensionless wave-number $k_1 y$ and whose convection velocity was nearly equal to the mean velocity at height y (see § 4 (*c*)), so it appears that the pressure fluctuations at wave-numbers above $k_1\delta_1 = 1$, are indeed generated by the 'universal' part of the inner layer turbulence. Wills's results in zero pressure gradient show that the convection velocity falls only to $0{\cdot}58U_1$ at the transducer cut-off wave-number, defined as $k_1 d = 2\pi$. $k\bar{y}_k$ is roughly 1 again but it very ill-defined because U varies only slowly (logarithmically) with y. Some further properties of the inner-layer contribution to the surface pressure fluctuation are discussed in § 4 (*d*).

The low-wave-number pressure fluctuations are generated outside the inner layer, at $y/\delta > 0{\cdot}2$ say. It is this latter part of the pressure-fluctuation field comprising virtually all the spectral density at low wave-number or low frequency, that contributes to the 'inactive' motion in the inner layer. At the lowest wave-numbers, at and below the point of maximum spectral density, the convection velocity $\overline{U}_c(k_1)$ decreases slightly but significantly (the same trend is shown by all four elements of the correlations, in-phase and quadrature, upstream and downstream). It seems likely that streamwise inhomogeneity of the boundary layer is in some way responsible. $k_1\delta_1 = 0{\cdot}3$ implies $\lambda = 20\delta_1 \simeq 5\delta$: δ increases by about 25 % in this distance but Wills's (1967) results indicate a similar trend in the boundary layer in zero pressure gradient where $d\delta/dx$ is less by a factor of four and dU_1/dx is of course zero. Whatever the explanation, the effect is certainly real as far as the application of the results to structural excitation is concerned.

The standard deviation σ of the spectral density, at constant wave-number, about the mean $\overline{U}_{\mathrm{ph}}$ does not vary greatly with wave-number. The standard deviation is not the same thing as the r.m.s. u-component fluctuation of the turbulence that produces the pressure fluctuations, but it is certainly of the same order of magnitude, and is less in the constant-pressure boundary layer than in the strongly retarded boundary layer by roughly the same factor as the r.m.s. turbulent intensity. The higher moments of the spectral density about the mean cannot be accurately determined from the experimental results, but it appears that the spectral density curves at constant wave-number are just significantly more peaked (higher kurtosis) than a Gaussian distribution at the highest and lowest wave-numbers.

(*b*) *The pressure–velocity correlations*

Wooldridge & Willmarth (1962) have measured correlations between the surface pressure fluctuation p_w and the u and v-component fluctuations with various separations in space and time. From these we have extracted the correlations with separation in the y direction only, which are plotted in figure 7: the correlation coefficients are always less than 0·1. The remarkable strength of the irrotational field in the retarded boundary layer with $U_1 \propto x^{-0.255}$ is shown most clearly



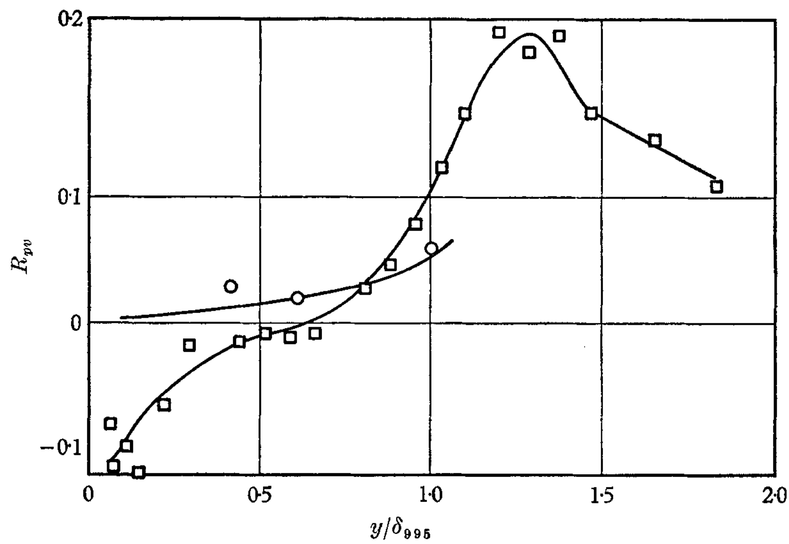


FIGURE 7. Coefficient of correlation between the velocity fluctuation at height y and the pressure fluctuation at the surface. (*a*) u-component, (*b*) v-component. ○, zero pressure gradient (Wooldridge & Willmarth 1962); □, $U_1 \propto x^{-0.255}$, $x = 48$ in.

by comparing measurements in this boundary layer with Wooldridge & Willmarth's results. Crudely speaking, a third of the u-component velocity fluctuation near the wall is correlated with the surface pressure fluctuation. The correlation decreases in magnitude at larger distances from the wall; the reason is *not* that the length scale of the pressure fluctuation is very small, but that most of the pressure fluctuation is generated in the region of maximum turbulent intensity near $y/\delta_1 = 2$, where it is more nearly in phase with $\partial u/\partial x$ than u: the correlation therefore passes through zero in this region.

The $\overline{p_w u}$ correlation for small y separation, measured in narrow frequency bands, is shown in figure 8 (a). As might be expected, the correlation coefficient rises to very high values at the lowest frequencies. There is a 'plateau' level of 0·2–0·25 in the medium-frequency range corresponding to the part of the pressure fluctuation that is generated in the inner layer. The filtered $\overline{p_w v}$ correlation near the surface (figure 8 (b)) is almost entirely confined to the very low frequencies: the maximum numerical value reached at the lowest frequencies is only about 0·15, so that the v-component fluctuation in the inner layer is much less affected by the pressure fluctuations than the u-component. This is a consequence of the continuity equation if the distance from the surface is small compared with typical wavelengths of the pressure fluctuations.

(c) *Convection velocity of the 'universal' inner layer motion*

Since $(\tau/\rho)^{\frac{1}{2}}$ is the only velocity scale of the inner layer turbulence, the convection velocity must be $U+c(\tau/\rho)^{\frac{1}{2}}$ where c is a constant depending on the precise definition of convection velocity. The simplest such definition is the energy-flux velocity $\overline{(U+u)q^2}/\overline{q^2}$, where $q^2 = u^2+v^2+w^2$, which combines advection of the turbulent kinetic energy $\frac{1}{2}\rho\overline{q^2}$ by the mean velocity field with transport of turbulent energy by the turbulent fluctuations themselves. We have not measured all three components of $\overline{q^2u}$ but measurements of $\overline{u^3}$ by Bradshaw (1967c) indicate that $\overline{u^3}/\overline{u^2} \simeq 0{\cdot}5(\tau/\rho)^{\frac{1}{2}}$ both in zero pressure gradient and with $U_1 \propto x^{-0{\cdot}255}$, and $\overline{q^2u}/\overline{q^2}$ will be of the same order. Thus $U_c \simeq U+0{\cdot}5(\tau/\rho)^{\frac{1}{2}}$, which is negligibly different from U for the purposes of the present paper. That $U_c > U$ although $\partial^2U/\partial y^2 < 0$ can be explained by the larger region of influence of eddies further from the wall.

(d) *Contribution of the 'universal' inner layer motion to the surface pressure fluctuation*

In §4 (a) the variation of $\overline{U}_{\mathrm{ph}}$ with k_1 was shown to be consistent with a 'universal' forcing function for the surface pressure fluctuation for $k_1\delta_1 > 1$ (or $k_1\delta_{995} > 4$ say) in the strongly retarded equilibrium boundary layer: the chief contributions at wave-number k_1 came from a distance roughly $1/k_1$ above the surface.

Taking τ = constant $= \tau_w$ for simplicity, we see that the spectral density of the surface pressure fluctuations due to the universal inner layer turbulence must be a universal function of τ and k_1 only, providing that the distance $1/k_1$ is large compared with the thickness of the sublayer, $12\nu/u_\tau$, but small compared with the thickness of the inner layer $\delta_i = 0{\cdot}2\,\delta$: in Wooldridge & Willmarth's (1962) boundary layer $u_\tau\delta/\nu$ was about 18 000 giving a ratio of 300 : 1 between the two thicknesses. This argument is very similar to Kolmogorov's argument (Batchelor 1953) that the spectrum of turbulence in the locally isotropic range depends only on the dissipation rate and the wave-number because no other length scale enters the problem. In the present case, dimensional arguments, or the substitution of 'universal' variables in the Poisson equation, show that the one-dimensional spectral density is proportional to τ_w^2/k_1 in the 'universal' range, whose centroid is somewhere near the geometric mean $k_1 = (u_\tau/12\nu\delta_i)^{\frac{1}{2}}$ or $k_1\delta_1 = 10$ in Wooldridge & Willmarth's experiment. The *two*-dimensional spectrum is proportional to $(\tau_w^2/k_1^2)f(k_3/k_1)$. Dimensional arguments also show that the surface pressure correlation should be constant at separations typical of the inner layer scales but as

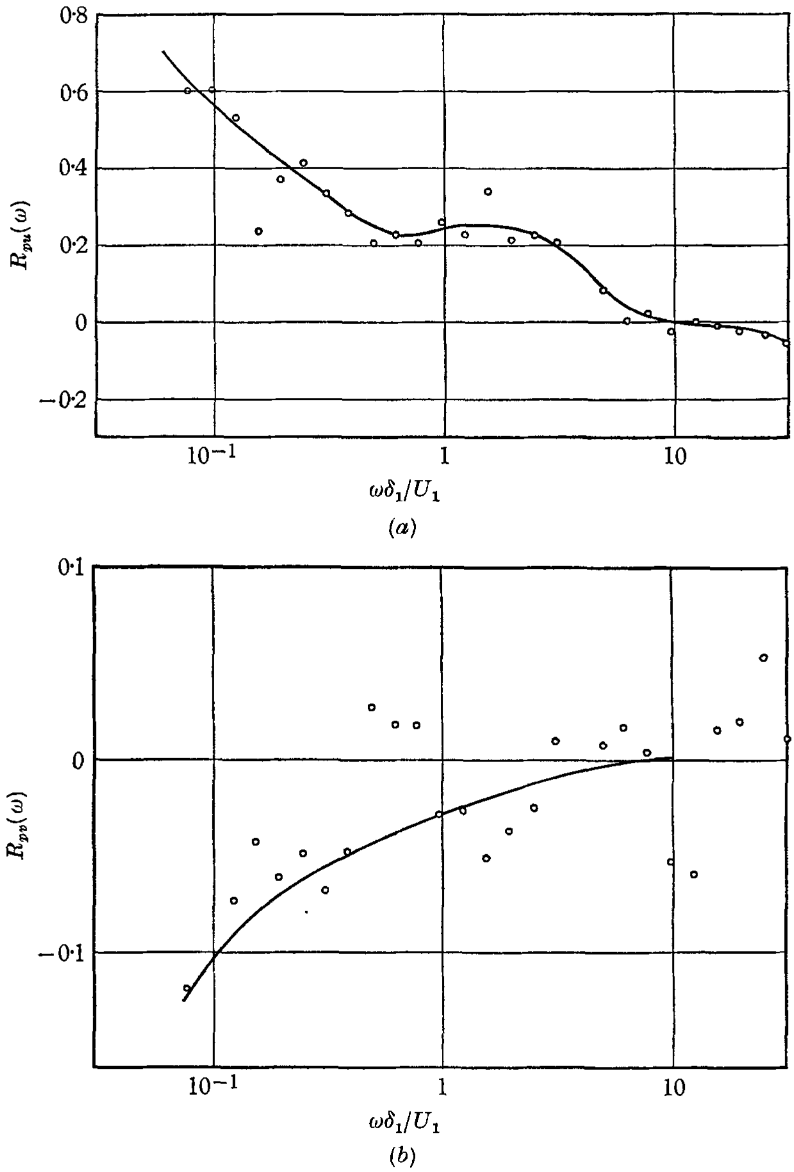


FIGURE 8. Coefficient of correlation between velocity fluctuation and surface pressure fluctuation in one-third octave frequency bands. $U_1 \propto x^{-0\cdot255}$, x = 48 in. (*a*) u-component, $y/\delta_{995} = 0\cdot037$; (*b*) v-component, $y/\delta_{995} = 0\cdot064$.

the correlation is the Fourier transform of the spectrum, which is an integral over *all* wave-numbers, the correlation behaviour will also depend on the behaviour of the spectrum at the ends of the k^{-1} range, unless the k^{-1} range contains nearly all the energy: the same applies to the Kolmogorov analysis for isotropic turbulence

(Webb 1964). The total energy in the k^{-1} range is τ_w^2 $(\log(u_\tau \delta_i/\nu)+\text{constant})$ so that according to the present analysis $\overline{p^2}/\tau_w^2$ should vary appreciably with Reynolds number.

These predictions for the surface pressure fluctuation seem to follow quite unambiguously from the hypothesis of 'universal' scaling of the inner layer motion, which is adequately confirmed by the results discussed in §2, but until recently the only direct experimental confirmation that the high-wave-number part of the surface pressure fluctuation is generated by the 'universal' inner layer motion was the behaviour of U_{ph} mentioned above. All published spectrum measurements showed a very rapid fall in spectral density in the region where a k^{-1} variation would be expected: figure 6 is typical. Recently, Hodgson (1967) has measured frequency spectra of surface pressure fluctuations using a probe microphone with an orifice of only 0·013 in. diameter. The spectra were found to vary roughly as $\omega^{-0\cdot 7}$ up to $\omega\delta_1/U_1 \simeq 10$ (say $\omega\delta_1/U_c \equiv k_1\delta_1 \simeq 15$). The index is expected to be less than unity because the convection velocity decreases as wave-number increases. Assuming ky_k = constant and noting that $U \propto y^{\frac{1}{6}}$ approximately over the inner part of the boundary layer (being exact where $U/u_\tau = 15$ if $K = 0\cdot 4$), we expect $\phi(\omega) \propto \omega^{-\frac{5}{6}}$. In view of the difficulty of measuring exact slopes, Hodgson's spectra strongly support the arguments leading to a k^{-1} spectrum. Moreover, the upper end of the power-law region corresponds to $u_\tau y_k/\nu \simeq 40$ which is where the mean velocity profile first deviates from the logarithmic law—that is, where the shear-producing, energy-containing eddies are first affected by viscosity. The large differences between Hodgson's results and previous measurements can be attributed to the inadequate spatial resolution of the transducers used by other workers. Corcos (1963, 1964) has derived correction formulae for spatial resolution, and his published correction of Willmarth's spectrum has a well-defined region of ω^{-1} variation in the range $1 < \omega\delta_1/U_1 < 10$ (say $1\cdot 5 < k\delta_1 < 15$), the transducer diameter being $0\cdot 3\delta_1$. If Willmarth & Roos's (1965) correction is used instead, the region of ω^{-1} variation ends at $\omega\delta_1/U_1 \simeq 5$, but in either case the increase in the mean-square intensity is considerable. Corcos's deduction from Willmarth & Roos's results that the contribution of the mean shear/turbulence term to the surface pressure is small for $u_\tau y/\nu < 100$ is also likely to be affected by inadequate resolution, for which no correction was made in this case: the transducer diameter was about $600\nu/u_\tau$.

Inner-layer universal scaling may be of help in deriving correction formulae for transducer resolution (Foxwell 1966) although the spatial correlations do not have such a simple form as the corresponding wave-number spectra. The most general correlation is the filtered space correlation, $\Gamma(\omega, \mathbf{r})$ in Corcos's (1964) notation, which is the double Fourier transform of the wave-number/frequency spectrum $\overline{p_w^2(\omega, k_1, k_3)}$. Noting that the inclusion of frequency, as well as wave-number, as a variable requires us to include the convection velocity at the given frequency also (and passing over the question of the precise definition of this convection velocity), we find (Bradshaw 1965) that

$$\overline{p_w^2(\omega, r_1, r_3)} \equiv \Gamma(\omega, \mathbf{r}) \sim \frac{1}{\omega} f\left(\frac{\omega r_1}{U_c}, \frac{r_3}{r_1}\right).$$

This Fourier transform is subject to the condition that the k^{-1} range contains nearly all the energy at the value of ω considered (a much less severe condition than the one mentioned above for the transform of $\overline{p_w^2(k_1)}$).

Further, Corcos (1964) and others have found empirically that

$$\Gamma(\omega, r_1) \sim \phi(\omega) f(\omega r_1/U_c)$$

and

$$\Gamma(\omega, r_3) = \phi(\omega) f(\omega r_3/U_c).$$

'Universal' scaling provides no information about the accuracy of Corcos's assumption of the separability of variables, leading to

$$\Gamma(\omega, \mathbf{r}) = \phi(\omega) A'(\omega r_1/U_c) B(\omega r_3/U_c).$$

However, it does imply that Willmarth & Roos's (1965) objection to Corcos's use of the more general form for $\Gamma(\omega, \mathbf{r})$ at high ω is not valid: Willmarth & Roos found experimentally that the attenuation of the pressure signal by the transducer was not a unique function of $\omega d/U_c$ (where d is the transducer diameter) as suggested by Corcos, but the universal scaling shows that it most certainly should be unique if the 'wave-number' ω/U_c is in the range in which the surface pressure fluctuation is dominated by contributions from the inner layer. Willmarth's extrapolation of his results to zero transducer diameter is not necessarily convincing. Foxwell (1966) presents results which collapse well on $\omega d/U_1$.

The practical conclusion to be drawn from this analysis, which is based on apparently unexceptionable deductions from the demonstration of universality of the active motion in the inner layer, is that the inaccuracies in pressure fluctuation measurements resulting from inadequate spatial resolution of the transducers are even more serious than at first appears. It does not necessarily follow that the unmeasurable part of the predicted spectrum is of any importance in determining structural excitation, because the wavelengths will be small compared with typical panel thickness.

Most of the experimental work reported here was done by Mr M. G. Terrell. The Fourier transform program used in the data reduction was written by Mr D. H. Ferriss and run on the KDF 9 computer of Mathematics Division, NPL. The pressure transducers were made under the direction of Mr W. W. Smith. I am indebted to Dr T. H. Hodgson of the College of Aeronautics, and Dr J. A. B. Wills, for helpful discussions, and to Professor G. M. Lilley of Southampton University, and Dr G. E. Gadd, for comments on a first draft of this paper.

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